§ 16.6 Applications of Surface Integrals

Review: Although they are defined invariantly (i.e. independent of coordinates/parameterization) line integrals and surface integrals can be evaluated in coordinates –

Line Integral: $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{v}(t)\,dt$

$\vec{r}(t) = (x(t), y(t), z(t))$

Surface Integral: $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$

$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

Stokes Theorem: $\displaystyle\iint_S \mathrm{Curl}\,\vec{F}\cdot\vec{n}\,dS = \int_C \vec{F}\cdot\vec{T}\,ds$

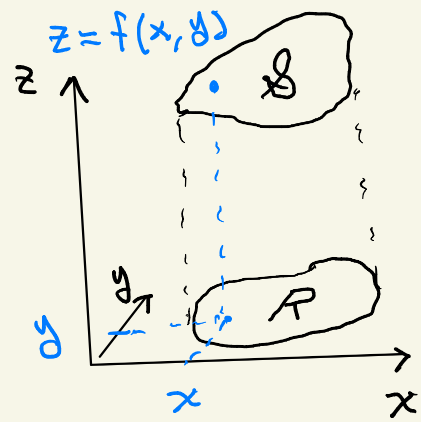
Surface Integral | Line Integral

Divergence Theorem: $\displaystyle\iiint_V \mathrm{Div}\,\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,dS$

Regular Triple Integral | Line Integral (Flux)

Example ①: What does the amplification factor for surface area $dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$ reduce to when $z = f(x,y)$ gives S and $(u,v) = (x,y)$?

Soln: Consider the special case when a surface is given $z = f(x,y)$.



In this case we can take $u = x$, $v = y$

Then: $\vec{r}(u,v) = \overrightarrow{(u, v, f(u,v))}$

Or use $\overrightarrow{(x,y)}$ for (u,v):

$$\vec{r}(x,y) = \overrightarrow{(x, y, f(x,y))}$$

Then:
$$\vec{r}_u = \vec{r}_x = \overrightarrow{(1, 0, f_x)}$$
$$\vec{r}_v = \vec{r}_y = \overrightarrow{(0, 1, f_y)}$$

$$\vec{r}_u \times \vec{r}_v = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \underset{\sim}{i}(-f_x) - \underset{\sim}{j}(f_y) + \underset{\sim}{k}$$
$$= -\overrightarrow{(f_x, f_y, -1)}$$

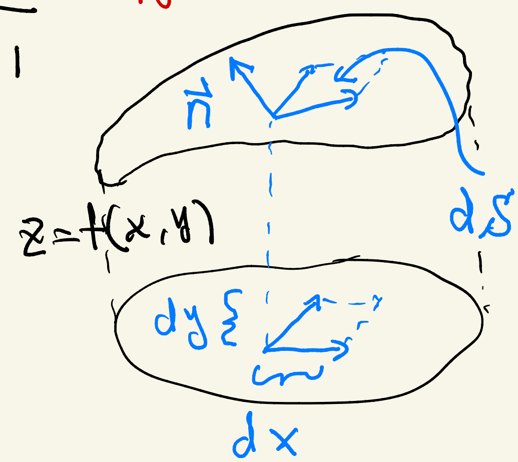
Amp. Factor for Area: $|\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1}$

Conclude:

$$dS = \sqrt{f_x^2 + f_y^2 + 1}\; dx\,dy$$

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} = \frac{\overrightarrow{(-f_x, -f_y, 1)}}{\sqrt{f_x^2 + f_y^2 + 1}}$$

pos z-component
⇒ upward normal
to surface



• **Note:** The z-component of the unit normal also supplies 1/(amplification factor):

Theorem: When $\vec{r}(x,y) = \overrightarrow{(x, y, f(x,y))}$ parameterizes a surface, $(x,y) \in R \subseteq \mathbb{R}^2$, the amplification factor for area is given by $(\vec{n} \cdot \underset{\sim}{k})^{-1}$, so

$$dS = \frac{1}{\vec{n} \cdot \underset{\sim}{k}}\, dx\,dy$$

Note: This is often the easiest way to find amplification factor since its easy to find the unit normal: Eg $F(x,y,z) = z - f(x,y) = 0$ gives $\mathcal{S}$ as the level set of F, ⇒ ∇F points normal to surface, so $\vec{n} = \frac{\nabla F}{|\nabla F|}$

Ex 2 Find the surface area of the paraboloid

$z = f(x,y) = x^2+y^2$, $(x,y)\in R = \{\underset{\sim}{x} : |\underset{\sim}{x}| < R\}$

Soln: Let S be the set of $\overrightarrow{(x,y,f(x,y))}$ for $(x,y)\in R$. This is level set of $F(x,y,z) = z - f(x,y)$.

$$\nabla F = \overrightarrow{(-f_x, -f_y, 1)} = \overrightarrow{(-2x,-2y,1)}$$

$$\vec{n} = \frac{\nabla F}{\|\nabla F\|}, \quad dS = A\,dx\,dy, \quad A = \frac{1}{\vec{n}\cdot \underset{\sim}{k}} = \sqrt{4x^2+4y^2+1}$$

$$\text{Area}(R) = \iint_R A\,dx\,dy = \iint_R \sqrt{4x^2+4y^2+1}\,dx\,dy$$

polar coordinates

$r^2 = x^2+y^2 \quad dA_{r\theta} = r\,dx\,dy$

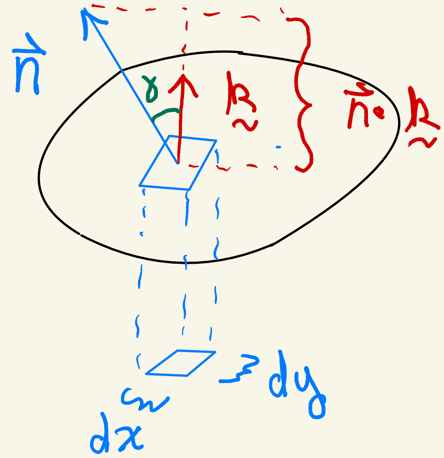
$$= \int_0^{2\pi}\int_0^R \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$u = 4r^2+1$, $du = 8r\,dr$

$$= \int_0^{2\pi}\int_{r=0}^{r=R} \sqrt{u}\;\frac{1}{8}\,du\,d\theta = \frac{2\pi}{8}\;\frac{2u^{3/2}}{3}\Bigg]_{r=0}^{r=R}$$

$$= \frac{\pi}{12}(4r^2+1)^{3/2}\Bigg]_0^R = \boxed{\frac{\pi}{12}\left[(4R^2+1)^{3/2}-1\right]}$$

Ex 3 Interpret the amplification factor geometrically when $\vec{r}(x,y) = \overrightarrow{(x, y, f(x,y))}$



Soln Let γ denote the angle between $\vec{n}$ and $\underset{\sim}{k}$. Then $\cos\gamma = \vec{n}\cdot\underset{\sim}{k}$. Therefore

$$dS = A\,dx\,dy \quad \text{where } A = \frac{1}{\cos\gamma}.$$

Summary: The general formula for surface area in general coord system $\vec{r}(u,v) = \overrightarrow{(x(u,v), y(u,v), z(u,v))}$ is

$$dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv \qquad (u,v)\in R$$

Surface Area: $\displaystyle\iint_R |\vec{r}_u \times \vec{r}_v|\,du\,dv$

Flux: $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot\vec{r}_u\times\vec{r}_v\,du\,dv$

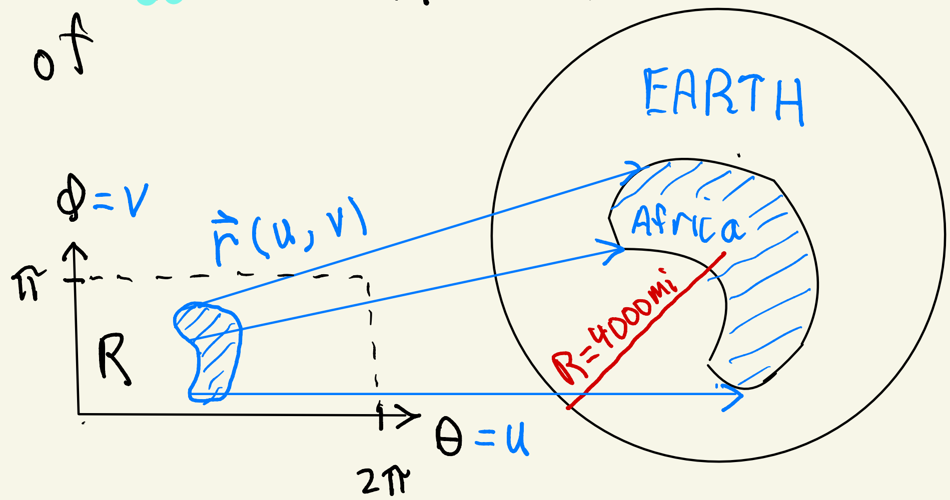
When $z = f(x,y)$: **Area** $= \displaystyle\iint_R \frac{1}{\vec{n}\cdot\underset{\sim}{k}}\,dx\,dy$

Flux: $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot\overrightarrow{(-f_x, -f_y, 1)}\,dx\,dy$

Finding areas on a map

R = "map"

Ex 4: Use the theory of integration on surfaces to explain how to find areas on a map –

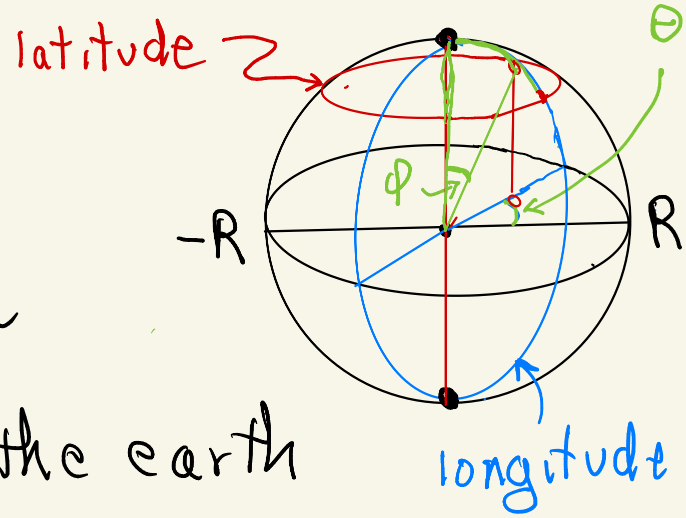


Soln: A "map" is a way assigining points on the earth to points on a sheet of paper. In theory then, a map of the earth is any coordinate system $\vec{r}(u,v) = \underbrace{(x(u,v), y(u,v), z(u,v))}_{\text{a point on Earth}}$

Maps always create a distortion of areas because the earth is curved and paper is flat. I.e., the area on the paper map will never scale up to area on Earth because there is always an amplification factor $A \neq 1$.

$$\underbrace{dS}_{\text{Area on Earth}} = A \underbrace{du\,dv}_{\text{Area on map}}$$

As the basic example - consider the map which uses latitude and longitude as the coordinates:



latitude $= \varphi$ $\quad 0 \le \varphi \le \pi$

longitude $= \theta$ $\quad 0 \le \theta \le 2\pi$

To describe our map of the earth

all we need is $\vec{r}(\varphi,\theta) = \overrightarrow{(x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta))}$

We get x, y, z from spherical coordinates:

Recall: $\overrightarrow{(\rho, \varphi, \theta)}$, $\rho = \sqrt{x^2+y^2+z^2}$ = dist to center $= R$

$$x = r\cos\theta = \rho\cos\theta\sin\varphi \qquad r = \rho\sin\varphi$$

$$y = r\sin\theta = \rho\sin\theta\sin\varphi \qquad z = \rho\cos\varphi$$

R = Radius of earth $\approx$ 4000 miles

$$\Rightarrow \quad x(\varphi,\theta) = R\cos\theta\sin\varphi$$

$$y(\varphi,\theta) = R\sin\theta\sin\varphi$$

Map $\Rightarrow$ $\vec{r}(\varphi,\theta) = R\overrightarrow{(\underbrace{\cos\theta\sin\varphi}_{x(u,v)}, \underbrace{\sin\theta\sin\varphi}_{y(u,v)}, \underbrace{\cos\varphi}_{z(u,v)})}$ $\qquad u = \varphi$, $v = \theta$

Conclude: $\vec{r}(u,v) = R\,\overrightarrow{(\cos u \sin v, \sin u \sin v, \cos v)}$

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = R\,\overrightarrow{(-\sin u \sin v, \cos u \sin v, 0)}$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = R\,\overrightarrow{(\cos u \cos v, \sin u \cos v, 0)}$$

$$dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -R\sin u \sin v & R\cos u \sin v & 0 \\ R\cos u \cos v & R\sin u \cos v & -R\sin v \end{vmatrix}$$

$$= \underset{\sim}{i}\left(-R^2 \cos u \sin^2 v\right) - \underset{\sim}{j}\left(R^2 \sin u \sin^2 v\right)$$

$$+ \underset{\sim}{k}\Big(-R^2 \underbrace{(\cos^2 u + \sin^2 u)}_{1} \cos v \sin v\Big)$$

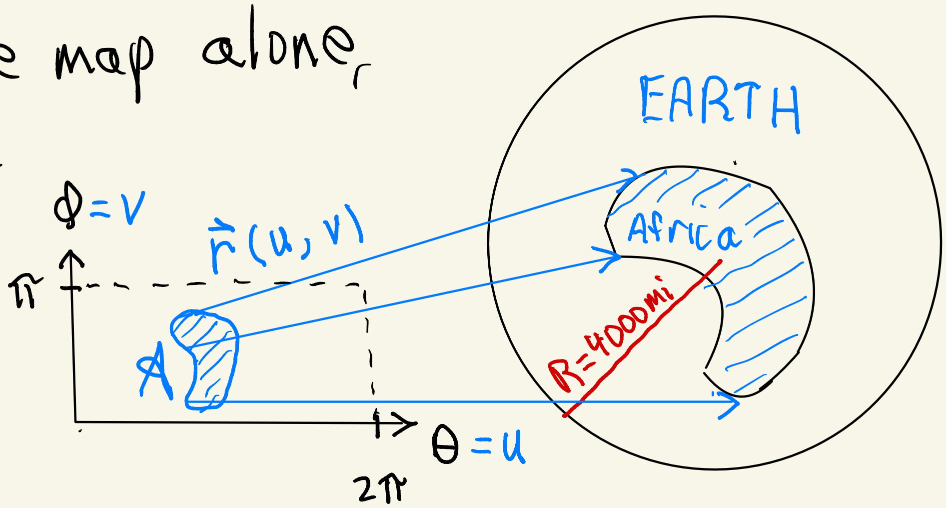
$$|\vec{r}_u \times \vec{r}_v|^2 = R^4\left(\cos^2 u \sin^4 v + \sin^2 u \sin^4 v + \cos^2 v \sin^2 v\right)$$

$$= R^4\left(\sin^4 v + \cos^2 v \sin^2 v\right) = R^4 \underbrace{(\sin^2 v + \cos^2 v)}_{1}(\sin^2 v)$$

$$= R^4 \sin^2 v$$

$$\Rightarrow \boxed{|\vec{r}_u \times \vec{r}_v| = R^2 \sin v}$$

Conclude: To find the area of Africa by using the map alone, just integrate over the picture of Africa on your map, using the amplification factor $dS = R^2 \sin u \, du\, dv$



Ie.

$$\text{Area of Africa} = R^2 \iint_{A} \sin u \, du\, dv$$

region corresponding to Africa on your map

Or in original variables:

$$\boxed{\text{Area of Africa} = R^2 \iint_{A} \sin\phi \, d\phi\, d\theta}$$